Revision Notes:
Relativity and Quantum Mechanics

0.1 Notation for Relativity

Define coordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. Consider a homogeneous Lorentz tranformation $(x^0, x^1, x^2, x^3) \rightarrow (x'^0, x'^1, x'^2, x'^3)$. This means any combination of velocity tranformations and rotations. A set of four quantities a^μ ($\mu = 0, 1, 2, 3$) tranforming according to the rule

$$a^\mu \longrightarrow a'^\mu = \frac{\partial x'^\mu}{\partial x^0} a^0 + \frac{\partial x'^\mu}{\partial x^1} a^1 + \frac{\partial x'^\mu}{\partial x^2} a^2 + \frac{\partial x'^\mu}{\partial x^3} a^3 \tag{1}$$

$$\equiv \frac{\partial x'^\mu}{\partial x^\nu} a^\nu \tag{2}$$

is called a **contravariant** 4-vector, written with an upper index. (Note the **summation convention** above — every index repeated on the same side of an equation is to be summed over, from 0 to 3.) Clearly x^μ is an example of a contravariant 4-vector.

There are also **covariant** 4-vectors, written with a lower index, which transform according to

$$a_\mu \longrightarrow a'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} a_\nu \, . \tag{3}$$

An obvious example is the vector operator $\partial_\mu = \partial / \partial x^\mu$.

The scalar product of a covariant and a contravariant 4-vector

$$a_\mu b^\mu \equiv a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3 \, , \tag{4}$$

is Lorentz invariant:

$$a'_\mu b'^\mu = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\lambda} a_\nu b^\lambda = \frac{\partial x^\nu}{\partial x^\lambda} a_\nu b^\lambda = \delta^\nu_\lambda a_\nu b^\lambda = a_\nu b^\nu \, , \tag{5}$$

where $\delta^\nu_\lambda = 1$ for $\nu = \lambda$, 0 for $\nu \neq \lambda$. But we know that $s^2 = c^2 t^2 - x^2 - y^2 - z^2$ is Lorentz invariant. We can write this as $s^2 = x_\mu x^\mu$ where

$$x_0 = ct, \quad x_1 = -x, \quad x_2 = -y \quad x_3 = -z \, . \tag{6}$$

x_μ is a covariant 4-vector formed from the contravariant 4-vectore x^μ by the operation

$$x_\mu = g_{\mu\nu} x^\nu \tag{7}$$

where the **metric tensor** $g_{\mu\nu}$ has all elements zero except the diagonal ones $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$. Thus we can make a covariant 4-vector from any contravariant one ("lower an index") by multiplying by the matrix $g_{\mu\nu}$. Similarly, we can "raise an index" with $g^{\mu\nu}$, which has identical components to $g_{\mu\nu}$:

$$a_\mu = g_{\mu\nu} a^\nu \, . \tag{8}$$

Note that

$$g^\mu{}_\nu = g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu \, . \tag{9}$$

Some important 4-vectors, in their contravariant form, are

- 4-momentum

$$p^\mu = (E/c, p_x, p_y, p_z) \tag{10}$$

- 4-momentum operator

$$i\hbar\partial^\mu = i\hbar g^{\mu\nu}\partial_\nu = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{-\partial}{\partial x}, \frac{-\partial}{\partial y}, \frac{-\partial}{\partial z}\right) \tag{11}$$

(note signs)

- 4-potential

$$A^\mu = (V/c, A_x, A_y, A_z)\,. \tag{12}$$

Lorentz transformations are usually written

$$a'^\mu \;=\; \Lambda^\mu{}_\nu a^\nu\,;\;\; \Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \tag{13}$$

$$a'_\mu \;=\; \Lambda_\mu{}^\nu a_\nu\,;\;\; \Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}\,. \tag{14}$$

You can check that

$$\Lambda_\mu{}^\nu = g_{\mu\lambda}g^{\nu\sigma}\Lambda^\lambda{}_\sigma \tag{15}$$

as expected. Lorentz transformations have the important property

$$\Lambda_\mu{}^\nu\Lambda^\mu{}_\lambda = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\lambda} = \delta^\nu{}_\lambda\,. \tag{16}$$

Hence

$$\Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu\,. \tag{17}$$

You can check this explicitly for a pure velocity transformation along the x-axis:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = (1 - v^2/c^2)^{-\frac{1}{2}}\,. \tag{18}$$

$(\Lambda^{-1})^\mu{}_\nu$ is the same expect $v \to -v$.

We can write

$$\Lambda^\mu{}_\nu = [\exp(\omega K_x)]^\mu{}_\nu \tag{19}$$

(which you can verify by expanding the exponential as a power series) where ω is the **rapidity**,

$$\omega = \tanh^{-1}(v/c)\,, \tag{20}$$

and K_x is the **generator** of velocity transformations (boosts) along the x-axis,

$$(K_x)^\mu{}_\nu = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{21}$$

For successive boosts in the same direction

$$\Lambda_1\Lambda_2 = \exp(\omega_1 K_x)\exp(\omega_2 K_x) = \exp[(\omega_1+\omega_2)K_x]\,, \tag{22}$$

so ω is additive.

To write the Dirac equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \beta mc^2\Psi - i\hbar c\vec{\alpha}\cdot\vec{\nabla}\Psi \tag{23}$$

in "covariant" notation we multiply on the left by β/c and rearrange terms to get

$$i\hbar\gamma^\mu\partial_\mu\Psi - mc\Psi = 0 \tag{24}$$

where

$$\gamma^0 = \beta\,,\;\; \gamma^j = \beta\alpha_j \;\; (j=1,2,3)\,. \tag{25}$$

If we need to use explicit matrices, we shall use those that follow from our choice for β and α_j in the lectures:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \tag{26}$$

where the "elements" are 2×2 submatrices, e. g. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The γ matrices have the property

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I \tag{27}$$

where I represents a 4×4 unit matrix (often omitted). Note that γ^μ **is not a 4-vector**. It is simply a set of four constant matrices, invariant under Lorentz transformations. Ψ has 4 components but it is neither an invariant nor a 4-vector — it is called a **spinor** and has special Lorentz transformation properties, which we shall not use in this course.

The **Feynman slash** notation is often used for brevity:

$$\not{a} \equiv \gamma^\mu a_\mu = g_{\mu\nu}\gamma^\mu a^\nu\,. \tag{28}$$

Explicitly

$$\not{a} = \begin{pmatrix} a^0 & 0 & -a^3 & -a^1+ia^2 \\ 0 & a^0 & -a^1-ia^2 & a^3 \\ a^3 & a^1-ia^2 & -a^0 & 0 \\ a^1+ia^2 & -a^3 & 0 & -a^0 \end{pmatrix}. \tag{29}$$

The Dirac equation is the

$$(i\hbar\not{\partial} - mc)\Psi = 0\,, \tag{30}$$

i. e.

$$(\not{p} - mc)\Psi = 0\,. \tag{31}$$

In practice we shall usually set $\hbar = c = 1$.

0.2 Transition Rates: Fermi's Golden Rule

Much of particle physics is about the calculation of decay rates and scattering cross sections. These are derived from quantum mechanical transition rates. Let us start by recalling how transition rates are obtained in non–relativistic quantum mechanics.

Suppose we have a Hamiltonian H_0 with eigenstates $\phi_n(\vec{r})$ normalized in some volume element V:

$$H_0\phi_n = E_n\phi_n\,, \quad \int_V \phi_m^*\phi_n d^3r = \delta_{mn}\,. \tag{32}$$

Consider some perturbation H':

$$(H_0 + H')\Psi = i\frac{\partial\Psi}{\partial t} \tag{33}$$

(remember that $\hbar = c = 1$). We want to know the transition rate to some state ϕ_f given that we start (say, at $t = -T/2$) in some state ϕ_i. We write

$$\phi(x) = \sum_n c_n(t)\phi_n(\vec{r})\, e^{-iE_n t} \tag{34}$$

(x represents the 4-vector $(t, \vec{r})$), where $c_n(-T/2) = \delta_{ni}$. We easily find

$$\frac{dc_f}{dt} = -i\sum_n c_n(t)\int d^3r\; \phi_f^* H'\phi_n\, e^{i(E_f-E_n)t} \tag{35}$$

$$\simeq -i\,\langle f|H'|i\rangle\, e^{i(E_f-E_i)t} \tag{36}$$

(assuming that the perturbation is small), where

$$\langle f|H'|i\rangle \equiv \int \phi_f^* H'\phi_i\, d^3r\,. \tag{37}$$

Hence

$$c_f(t) \simeq -i\int_{-T/2}^{t} dt'\,\langle f|H'|i\rangle\, e^{i(E_f-E_i)t'}\,. \tag{38}$$

The transition amplitude (in the far future, $t = +T/2$) is thus

$$A_{fi} = c_f(+T/2) = -i\int_{-T/2}^{+T/2} dt\,\langle f|H'|i\rangle\, e^{i(E_f-E_i)t}\,. \tag{39}$$

We can write in covariant notation

$$\lim_{T\to\infty} A_{fi} = -i\int \phi_f^*(x)H'\phi_i(x)\, d^4x \tag{40}$$

where

$$\phi_n(x) = \phi_n(\vec{r})e^{-iE_n t}\,. \tag{41}$$

If H' is time-dependent we have a transition probability

$$\lim_{T\to\infty}|A_{fi}|^2 = |\langle f|H'|i\rangle|^2 \int_{-T/2}^{+T/2} dt\, e^{i(E_f-E_i)t}\int_{-T/2}^{+T/2} dt'\, e^{i(E_f-E_i)t'} \tag{42}$$

$$= 2\pi\,|\langle f|H'|i\rangle|^2\,\delta(E_f - E_i)\,T\,. \tag{43}$$

Thus the transition rate is

$$\Gamma(i \to f) = \lim_{T\to\infty} \frac{|A_{fi}|^2}{T} = 2\pi \, |\langle f|H'|i\rangle|^2 \, \delta(E_f - E_i) \, . \tag{44}$$

If we want to integrate over a number of possible final states with density $\rho(E_f)$ around energy E_f, we get

$$\begin{aligned} \Gamma(i \to f) &= \lim_{T\to\infty} \frac{1}{T} \int |A_{fi}|^2 \rho(E_f) \, dE_f & (45) \\ &= 2\pi \, |\langle f|H'|i\rangle|^2 \, \rho(E_i) \, , & (46) \end{aligned}$$

which is **Fermi's Golden Rule**.

We can obtain the next correction by successive substitution:

$$\begin{aligned} \frac{dc_f}{dt} &\simeq -i \, \langle f|H'|i\rangle \, e^{i(E_f - E_i)t} & (47) \\ &\quad +(-i)^2 \sum_{n\neq i} \langle f|H'|n\rangle \, e^{i(E_f-E_n)t} \int_{-T/2}^{t} dt' \, \langle n|H'|i\rangle \, e^{i(E_n-E_i)t'} \, . & (48) \end{aligned}$$

Since we are assuming the perturbation was not present at $t = -T/2$ but was constant after that, we should interpret

$$\int_{-T/2}^{t} dt' \, \langle n|H'|i\rangle \, e^{i(E_n-E_i)t'} = \langle n|H'|i\rangle \, \frac{e^{i(E_n-E_i)t}}{i(E_n - E_i)} \, , \tag{49}$$

so

$$\frac{dc_f}{dt} = -ie^{i(E_f-E_i)t} \left[\langle f|H'|i\rangle + \sum_{n\neq i} \frac{\langle f|H'|n\rangle \, \langle n|H'|i\rangle}{E_i - E_n} + \ldots \right] \, . \tag{50}$$

Then Fermi's Golden Rule becomes

$$\Gamma(i \to f) = 2\pi \, |T_{fi}|^2 \, \rho(E_i) \tag{51}$$

where

$$T_{fi} = \langle f|H'|i\rangle + \sum_{n\neq i} \frac{\langle f|H'|n\rangle \, \langle n|H'|i\rangle}{E_i - E_n} + \ldots \, . \tag{52}$$

Problem 1:
By further successive substitution, find the next (i. e. third–order) term in equation (52).

0.3 Phase Space

Consider now the transition rate for the general decay process $a \to 1 + 2 + 3 + \ldots + n$. There are $(n-1)$ independent momenta in the final state (because $\vec{p}_1 + \ldots + \vec{p}_n = \vec{p}_a$) and if all wavefunctions are normalized to one particle per unit volume there is one per volume h^3 of momentum space, i.e. one per $(2\pi)^3$ volume since $\hbar = 1$ implies $h = 2\pi$. Therefore the total decay rate per initial particle is

$$\begin{aligned}
\Gamma &= 2\pi \int \frac{d^3\vec{p}_1}{(2\pi)^3} \cdots \frac{d^3\vec{p}_{n-1}}{(2\pi)^3} |T_{fi}|^2 \delta \left(E_a - \sum_{j=1}^{n} E_j \right) && (53) \\
&= (2\pi)^{4-3n} \int d^3\vec{p}_1 \ldots d^3\vec{p}_n \, |T_{fi}|^2 \, \delta^3 \left(\vec{p}_a - \sum \vec{p}_j \right) \, \delta \left(E_a - \sum E_j \right) . && (54)
\end{aligned}$$

However, normalizing to one particle per unit volume is not a Lorentz invariant procedure: it is only true in one frame since volume elements are Lorentz contracted (the particle density increased by γ) in other frames. Now the density is the timelike component of a 4-vector, transforming like E, so a relativistic normalization should be proportional to E particles per unit volume. The usual convention is to normalize to $2E$ particles per unit volume (the reason will appear shortly). The corresponding invariant matrix element for $a \to 1 + 2 + \ldots + n$ is then

$$M_{fi} = (2E_a \cdot 2E_1 \cdots 2E_n)^{1/2} T_{fi} , \tag{55}$$

and

$$\Gamma = \frac{(2\pi)^{4-3n}}{2E_a} \int \frac{d^3\vec{p}_1}{2E_1} \cdots \frac{d^3\vec{p}_n}{2E_n} |M_{fi}|^2 \, \delta^3 \left(\vec{p}_a - \sum \vec{p}_j \right) \, \delta \left(E_a - \sum E_j \right) . \tag{56}$$

Now $E_j = (\vec{p}_j^{\,2} + m_j^2)^{1/2}$ so inside the integral we can write

$$\frac{d^3\vec{p}_j}{2E_j} = d^3\vec{p}_j \, dE_j \, \delta(p_j^\mu p_{j\,\mu} - m_j^2) . \tag{57}$$

This is Lorentz invariant so the integral is now frame–independent. Γ is proportional to E_a^{-1} due to the time–dilatation of lifetime: $\tau_a = \Gamma^{-1} \sim E_a$. The integral in (56) is called a **phase-space** integral.

We normalize to $2E$ particles because of the simple relation (57), which follows from the useful general relation

$$\int dE \, \delta[f(E)] = 1 \bigg/ \left| \frac{df}{dE} \right|_{f(E)=0} . \tag{58}$$

0.4 Two-body Decay

Consider the decay $a \to b + c$ in the rest–frame of a, where

$$p_a^\mu = (E_a, \vec{p}_a) = (m_a, 0) . \tag{59}$$

Equation (56) gives

$$\begin{aligned}
\Gamma &= \frac{(2\pi)^{-2}}{2m_a} \int \frac{d^3\vec{p}_b}{2E_b} \frac{d^3\vec{p}_c}{2E_c} |M_{fi}|^2 \, \delta^3(\vec{p}_b + \vec{p}_c) \, \delta(m_a - E_b - E_c) && (60) \\
&= \frac{(2\pi)^{-2}}{2m_a} \int \frac{d^3\vec{p}_b}{4E_b E_c} |M_{fi}|^2 \, \delta(m_a - E_b - E_c) . && (61)
\end{aligned}$$

We can write $d^3\vec{p}_b = p_b^2\, dp_b\, \sin\theta\, d\theta d\phi$. Also

$$E_b = (p_b^2 + m_b^2)^{1/2}\,, \quad E_c = (p_b^2 + m_c^2)^{1/2} \tag{62}$$

since $\vec{p_c} = -\vec{p}_b$. Now

$$\begin{aligned} \int dp_b\, \delta\left[m_a^2 - (p_b^2 + m_b^2)^{1/2} - (p_b^2 + m_c^2)^{1/2}\right] &= \left[\frac{p_b}{(p_b^2 + m_b^2)^{1/2}} + \frac{p_b}{(p_b^2 + m_c^2)^{1/2}}\right]^{-1} \\ &= \frac{E_b E_c}{m_a p_b}\,, \end{aligned} \tag{63}$$

where we used eq. (58) with p_b in the place of E. Hence

$$\Gamma = \frac{p_b}{32\pi^2 m_a^2} \int |M_{fi}|^2\, \sin\theta d\theta d\phi\,. \tag{64}$$

If $|M_{fi}|^2$ is independent of the decay angles θ and ϕ, then it is just a number and

$$\Gamma(a \to b + c) = \frac{p_b}{8\pi m_a^2}\, |M_{fi}|^2\,. \tag{65}$$

Remember that p_b here means the 3-momentum of b in the rest frame of a.

Problem 2:
Show that

$$p_b = \left[(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(m_a - m_b - m_c)\right]^{1/2} / (2m_a)\,. \tag{66}$$

0.5 Two-body Scattering

We can also use Fermi's Golden Rule to calculate the transition rate for a scattering process such as $a + b \to c + d$. The invariant matrix element will again be normalized to $2E$ particles per unit volume, so

$$M_{fi} = (2E_a \cdot 2E_b \cdot 2E_c \cdot 2E_d)^{1/2} T_{fi}\,, \tag{67}$$

$$\begin{aligned} \Gamma(a + b \to c + d) &= \frac{(2\pi)^{-2}}{2E_a\, 2E_b} \int \frac{d^3\vec{p}_c}{2E_c} \frac{d^3\vec{p}_d}{2E_d}\, |M_{fi}|^2\, \times \\ &\qquad \times \delta^3(\vec{p}_a + \vec{p}_b - \vec{p}_c - \vec{p}_d)\, \delta(E_a + E_b - E_c - E_d)\,. \end{aligned} \tag{68}$$

The integral is invariant; we choose to calculate it in the c. m. frame, where $\vec{p}_a = -\vec{p}_b$. Then the integral is the same as for two–body decay, with $\sqrt{s} = E + a + E_b$ in the place of m_a:

$$\text{Integral} = \frac{p_c^*}{4\sqrt{s}} \int |M_{fi}|^2\, d\Omega^*\,. \tag{69}$$

From now on in case of ambiguity we shall put a star on quantities defined in the c. m. frame; $d\Omega^*$ is the element of solid angle, $d\Omega^* = \sin\theta^* d\theta^* d\phi^*$.

We are interested in the cross section σ rather than the rate. It is defined in terms of the following quantities in the lab (rest frame of b):

$$\Gamma = (\text{Flux of } a) \times (\text{Density of } b) \times \sigma \,. \tag{70}$$

Remember Γ is defined in terms of T_{fi}, i. e. for unit density. Hence the flux of a is v_a in the lab frame, i. e. p_a/E_a. Also $E_b = m_b$ in the lab, so

$$\begin{aligned} \sigma(ab \to cd) &= \frac{E_a}{p_a}\frac{(2\pi)^{-2}}{4E_a m_b}\frac{p_c^*}{4\sqrt{s}}\int |M_{fi}|^2\, d\Omega^* \\ &= \frac{p_c^*}{64\pi^2 p_a m_b \sqrt{s}}\int |M_{fi}|^2\, d\Omega^* \,. \end{aligned} \tag{71}$$

Remember that p_a is the 3-momentum of a in the lab while p_c^* is that of c in the c. m. frame.

Problem 3:
Show that the lab and c. m. 3-momenta of particle a are related by

$$p_a m_b = p_a^* \sqrt{s}\,. \tag{72}$$

Using the results of problem 3 we may write the differential cross section in the c. m. frame as

$$\frac{d\sigma}{d\Omega^*}(ab \to cd) = \frac{1}{64\pi^2 s}\left(\frac{p_c^*}{p_a^*}\right)|M_{fi}|^2 \,. \tag{73}$$

The differential cross section is also often expressed in terms of the invariant 4-momentum transfer squared t (sometimes loosely referred to as just the momentum transfer)

$$t \equiv (p_c - p_a)^2 = m_a^2 + m_c^2 - 2p_a \cdot p_c \,, \tag{74}$$

where from now on p_a etc. refer to 4-momenta, so that $p_a^2 \equiv p_{a\mu}p_a^\mu = m_a^2$, $p_a \cdot p_c \equiv p_{a\mu}p_c^\mu$ etc.

In the c. m. frame, choosing the z axis along $\vec{p}_a^*$ and $\vec{p}_c^*$ in the x-z plane:

$$p_a^\mu = (E_a^*, 0, 0, p_a^*)\,, \tag{75}$$

$$p_c^* = (E_c^*, p_c^* \sin\theta^*, 0, p_c^* \cos\theta^*)\,, \tag{76}$$

so

$$p_a \cdot p_c = E_a^* E_c^* - p_a^* p_c^* \cos\theta^* \tag{77}$$

and

$$dt = -2p_a^* p_c^* \sin\theta^* d\theta^* \,. \tag{78}$$

Assuming no ϕ^* dependence of $|M_{fi}|^2$, we can write $d\Omega^* = -2\pi \sin\theta^* d\theta^*$. Hence

$$\frac{d\sigma}{dt}(ab \to cd) = \frac{1}{64\pi s (p_a^*)^2}|M_{fi}|^2 \,. \tag{79}$$

In addition to $s = (p_a + p_b)^2 = (p_c + p_d)^2$ and $t = (p_c - p_a)^2 = (p_b - p_d)^2$, another commonly–encountered invariant for the scattering process $a + b \to c + d$ is

$$u \equiv (p_a - p_d)^2 = (p_b - p_c)^2 \,. \tag{80}$$

The quantities s, t and u are called the **Mandelstam variables**.

Problem 4:
Show that the three Mandelstam variables are not independent but satisfy the equation

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \,. \tag{81}$$

0.6 Interaction via Particle Exchange

In particle physics we regard all forces as arising from particle exchange (exchange of quanta oof the interaction field). This is really just a way of looking at the terms in the perturbation theory expansion. Consider the shift in energy of the state $|i\rangle$ due to the interaction term H' in the Hamiltonian:

$$\Delta E_i = \langle i|H'|i\rangle + \sum_{j\neq i} \frac{\langle i|H'|j\rangle\langle j|H'|i\rangle}{E_i - E_j} + \ldots \,. \tag{82}$$

Suppose H' can cause emission or absorption of particles of rest–mass m. Bythis we mean that if $|i\rangle$ contains a point source of strength g at $\vec{r} = \vec{r}_1$ and $|j\rangle$ contains the source plus a particle of momentum $\vec{k}(=\hbar\vec{k})$, i. e. with wavefunction $\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$ (normalized to one particle per unit volume), then the contribution from particle emission to $\langle j|H'|i\rangle$ is

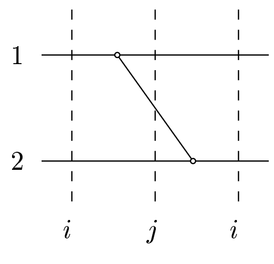
$$\frac{g}{\sqrt{eE_k}} \int d^3\vec{r}\phi^*(\vec{r})\delta^3(\vec{r} - \vec{r}_1) = \frac{g}{\sqrt{eE_k}}\, e^{-i\vec{k}\cdot\vec{r}_1} \tag{83}$$

where $E_k = (\vec{k}^2 + m^2)^{1/2}$. (N. B. g gives the invariant matrix element, normalized to $2E_k$ particles per unit volume, so the normalization factor must be devided out).

Similarly for absorption of the particle by a source at $\vec{r}_2$ we have a contribution to $\langle i|H'|j\rangle$ of $\frac{g}{\sqrt{eE_k}} e^{+i\vec{k}\cdot\vec{r}_2}$. Therefore exchange of the particles from source 1 to source 2 gives a contribution to ΔE_i, via the second term in the expansion (82), of

$$\Delta E_i^{1\to 2} = \widetilde{\sum_j} \frac{g^2}{2E_k} \frac{e^{-i\vec{k}\cdot(\vec{r}_2 - \vec{r}_1)}}{E_i - E_j} \,, \tag{84}$$

which can be represented by the diagram:



The intermediate state j consists of the sources plus the particle, so $E_j = E_i + E_k$. Note that the actual production of this state would violate energy conservation. It is a **virtual state** and the exchanged object is a **virtual paricle**. The diagram should not be taken too literally. In only depicts a contribution in the perturbation expansion.

The sum $\widetilde{\sum}$ represents a phase space integration over all momenta $\vec{k}$ of the exchanged particle, with (as usual) one state per $(2\pi)^3$ of momentum space. Therefore

$$\Delta E_i^{1\to 2} = \frac{g^2}{(2\pi)^3} \int \frac{d^3\vec{k}}{2E_k} \frac{e^{i\vec{k}\cdot(\vec{r}_2-\vec{r}_1)}}{-E_k} \tag{85}$$

$$= -\frac{g^2}{2(2\pi)^3} \int d^3\vec{k}\, \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+m^2} \qquad (\vec{r} \equiv \vec{r}_2 - \vec{r}_1)\,. \tag{86}$$

To do the integral choose the z axis along $\vec{r}$. Then $\vec{k}\cdot\vec{r} = kr\cos\theta$ and $d^3\vec{k}$ becomes $2\pi k^2\,dk\,d(\cos\theta)$, and the $\cos\theta$ integration gives

$$\Delta E_i^{1\to 2} = -\frac{g^2}{2(2\pi)^3} \int_0^\infty \frac{k^2 dk}{k^2+m^2} \frac{e^{ikr}-e^{-ikr}}{ikr}\,. \tag{87}$$

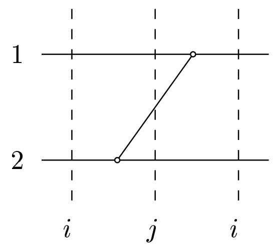
Write this integral as one half of the integral from $-\infty$ to ∞, which can be done by residues:

$$\Delta E_i^{1\to 2} = \frac{-g^2}{8\pi} \frac{e^{-mr}}{r}\,. \tag{88}$$

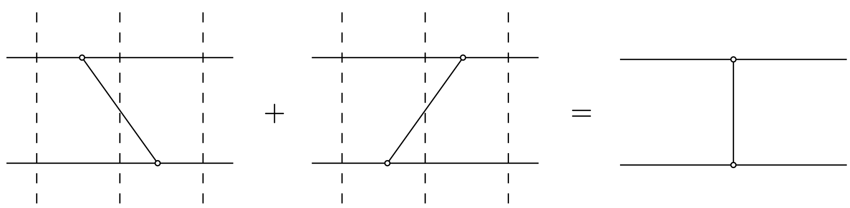
The contribution from emission from source 2 and absorption by 1 turns out to be the same:

$$\Delta E_i^{2\to 1} = \frac{-g^2}{8\pi} \frac{e^{-mr}}{r}\,. \tag{89}$$

It is represented by the diagram



These diagrams are called **time-ordered** (or **old-fashioned**) perturbation theory diagrams. The sum of all time orderings is represented by a **Feynman** diagram (or graph):



Because the intermediate state is virtual, the time ordering of emission and absorption is frame dependent, but the sum of all orderings (the Feynman graph) is frame independent:

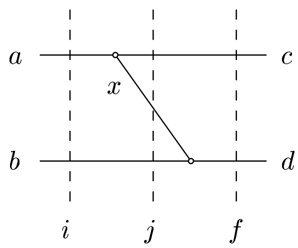
$$\Delta E_i = \frac{-g^2}{4\pi} \frac{e^{-mr}}{r} . \tag{90}$$

This is the **Yukawa potential**, due to single particle exchange. The exponential decrease has range $R = m^{-1}$, i. e. $R = \hbar/(mc)$, the Compton wavelength of the exchanged particle. In electromagnetism we have zero–mass photon exchange and hence "infinite range", $R = \infty$. In this case the Yukawa formula (90) reduces to the Coulomb potential.

0.7 Scattering via One-Particle Exchange

We can use the same method as for the Yukawa potential to find the differential cross section for the scattering process $a + b \to c + d$ via exchange of particle x. Instead of potential energy of two point sources, we now want the invariant matrix element M_{fi} where $|i\rangle$ consists of a and b with momenta $\vec{p}_a$ and $\vec{p}_b$ and $|f\rangle$ is $c + d$ with momenta $\vec{p}_c$, $\vec{p}_d$.

Consider first the contribution from the time ordering $a \to c + x$, $x + b \to d$:



The corresponding term in the perturbation expansion (52) of the non–invariant transition matrix element T_{fi} is

$$T_{fi} = \frac{\langle f|H'|j\rangle \, \langle j|H'|i\rangle}{E_i - E_j} , \tag{91}$$

i. e.

$$T_{fi}^{a\to b} = \frac{\langle d|H'|x+b\rangle \, \langle c+x|H'|a\rangle}{(E_a+E_b) - (E_c+E_x+E_d)} . \tag{92}$$

Notice that the momentum of x is fixed by $\vec{p}_x = \vec{p}_a - \vec{p}_c$ so there is no phase space integration. If the invariant matrix element for $a \to c + x$ is g_a, we have as usual

$$\langle c+x|H'|a\rangle = \frac{g_a}{(2E_a \, 2E_x \, 2E_c)^{1/2}} . \tag{93}$$

Similarly, define

$$\langle d|H'|x+b\rangle = \frac{g_b}{(2E_b \cdot 2E_x \cdot 2E_d)^{1/2}} . \tag{94}$$

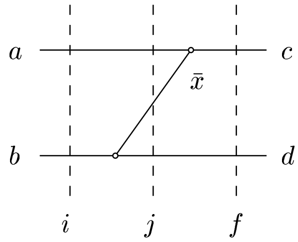
Then

$$M_{fi} = (2E_a \cdot 2E_b \cdot 2E_c \cdot 2E_d)^{1/2} T_{fi} , \tag{95}$$

giving

$$M_{fi}^{a\to b} = \frac{1}{2E_x} \frac{g_a g_b}{E_a - E_c - E_x} \,. \tag{96}$$

For the other time ordering



the quantum numbers are such that the exchanged particle must be $\bar{x}$, the antiparticle of x. For example, for $\bar{p}p \to \bar{n}n$ we could have $x = \pi^-$ and then $\bar{x} = \pi^+$. We assume **crossing symmetry**

$$\langle c|H'|a + \bar{x}\rangle = \langle c + x|H'|a\rangle \,, \quad \text{etc.} \tag{97}$$

Then

$$M_{fi}^{b\to a} = \frac{1}{2E_{\bar{x}}} \frac{g_a g_b}{E_b - E_d - E_{\bar{x}}} \,. \tag{98}$$

But $\vec{p}_{\bar{x}} = \vec{p}_b - \vec{p}_d$ and $\vec{p}_a + \vec{p}_b = \vec{p}_c + \vec{p}_d$, so $\vec{p}_{\bar{x}} = \vec{p}_c - \vec{p}_a = -\vec{p}_x$ and

$$E_{\bar{x}} = E_x = \left[(\vec{p}_a - \vec{p}_c)^2 + m_x^2\right]^{1/2} \,. \tag{99}$$

The

$$M_{fi} = M_{fi}^{a\to b} + M_{fi}^{a\to b} \tag{100}$$

$$= \frac{g_a g_b}{2E_x}\left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x}\right) \tag{101}$$

$$= \frac{g_a g_b}{2E_x}\left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_a - E_c + E_x}\right) \,, \tag{102}$$

since $E_a + E_b = E_c + E_d$. Combining the two terms gives

$$M_{fi} = \frac{g_a g_b}{2E_x} \frac{2E_x}{(E_a - E_c)^2 - E_x^2} \tag{103}$$

$$= \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_x^2} \tag{104}$$

$$= \frac{g_a g_b}{t - m_x^2} \,, \tag{105}$$

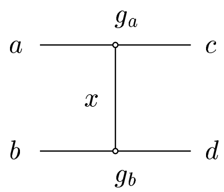
where t is the 4-momentum transfer squared, $(p_a - p_c)^2$, which is negative for the processes we shall encounter, so no infinity occurs in the differential cross section. Using our previous result (79), we have

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s (p_a^*)^2} \frac{g_a^2 g_b^2}{(t - m_x^2)^2} \,, \tag{106}$$

assuming that $g_{a,b}$ are real. The differential cross section has a forward ($t = 0$) peak with width of order m_x^2, corresponding to the range of interaction m_x^{-1}.

0.8 Feynman Graphs

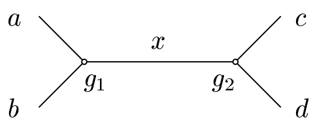
As in the calculation of the Yukawa potential, the sum of the time orderings, represented by a single Feynman graph,



$$M_{fi} = \frac{g_a g_b}{(p_a - p_c)^2 - m_x^2} \,, \tag{107}$$

has a simpler form than either individual term. For particles without spin, there is a **coupling constant** $g_{a,b}$ for each vertex and a **propagator** $(q^2 - m^2)^{-1}$ for each internal line of 4-momentum q^μ and mass (i. e. rest–mass) m. Notice that in Feynman graphs (unlike the old–fashioned, time–ordered graphs) 4-momentum is conserved at the vertices but internal lines are not constrained to have $q^2 = m^2$ as real particles must. These lines represent both a virtual particle going one way and a virtual antiparticle going the other. They are said to be **off mass shell** when $q^2 \neq m^2$ because the surface in 4-momentum space described by $q^\mu q_\mu = m^2$ (on which real particles lie) is called the mass shell.

Problem 5:
Using old–fashioned perturbation theory, verify that the invariant matrix element due to the Feynman graph



is

$$M_{fi} = \frac{g_1 g_2}{s - m_x^2} \,. \tag{108}$$

(Hint: Don't forget to include all time–orderings.)