Viscosity and scale invariance in the unitary Fermi gas

Searching for the perfect fluid

Tilman Enss (TU München)

Nancy, 7 February 2011

[Annals of Physics 326, 770 (2011); arXiv:1008.0007]

Technische Universität München

Outline

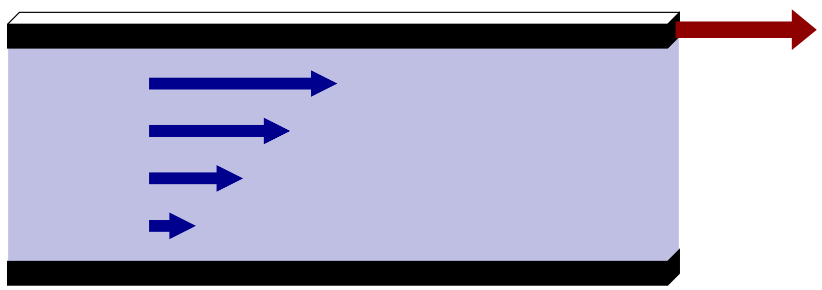
- shear viscosity and perfect fluidity
- transport equations for the unitary Fermi gas
- analytical and numerical results, universal high-frequency behavior
- scale invariance and universal dynamics

Technische Universität München

Shear viscosity: definition

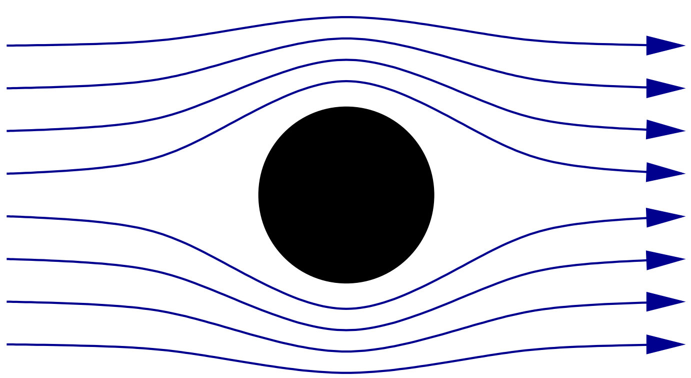
[Schaefer, Teaney, Rep. Progr. Phys. 2009]

- Viscosity determines shear stress, or friction, of fluid flow:



$$F = A\,\eta\,\frac{\partial v_x}{\partial y}$$

- dimensionless measure of shear stress: Reynolds number



$$Re = \underset{\text{fluid property}}{\frac{n}{\eta}} \times \underset{\text{flow property}}{mvr}$$

- relativistic systems: $Re = \dfrac{s}{\eta} \times \tau T$

Technische Universität München

Shear viscosity: estimates

- kinetic theory (Boltzmann equation) for dilute gas:

$$\eta = \tfrac{1}{3} n\, \bar{p}\, \ell_{\rm mfp}\,, \quad \ell_{\rm mfp} = \frac{1}{n\sigma} \;:\quad \eta \simeq \frac{\sqrt{m k_B T}}{\sigma(T)}$$

grows with T

- superfluid: $\eta_{\rm SF} = 0$

  but phonon contribution $\eta \sim T^{-5}$ [Landau, Khalatnikov 1949]

- **minimum in between:** at which temperature and viscosity?

- uncertainty suggests $\dfrac{\eta}{n} \sim \bar{p}\, \ell_{\rm mfp} \geq \hbar$ (careful!)

Insights from string theory

- conformal field theory (CFT) dual to AdS_5 black hole:

shear viscosity $\longleftrightarrow$ graviton absorption cross section (~ area of event horizon)

CFT entropy $\longleftrightarrow$ Hawking-Bekenstein entropy (~ area of event horizon)

- specifically SU(N), $\mathcal{N} = 4$ SYM theory (no confinement, no running cpl) in strong-coupling 't Hooft limit $\lambda = g^2 N$ is dual to classical gravity:

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

[Policastro, Son, Starinets 2001; Kovtun, Son, Starinets 2005]

- conjecture of **universal lower bound**

Technische Universität München

Perfect fluidity

[Schaefer, Teaney, Rep. Progr. Phys. 2009]

- Definition: **"perfect fluid"** saturates bound $$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

How to qualify?

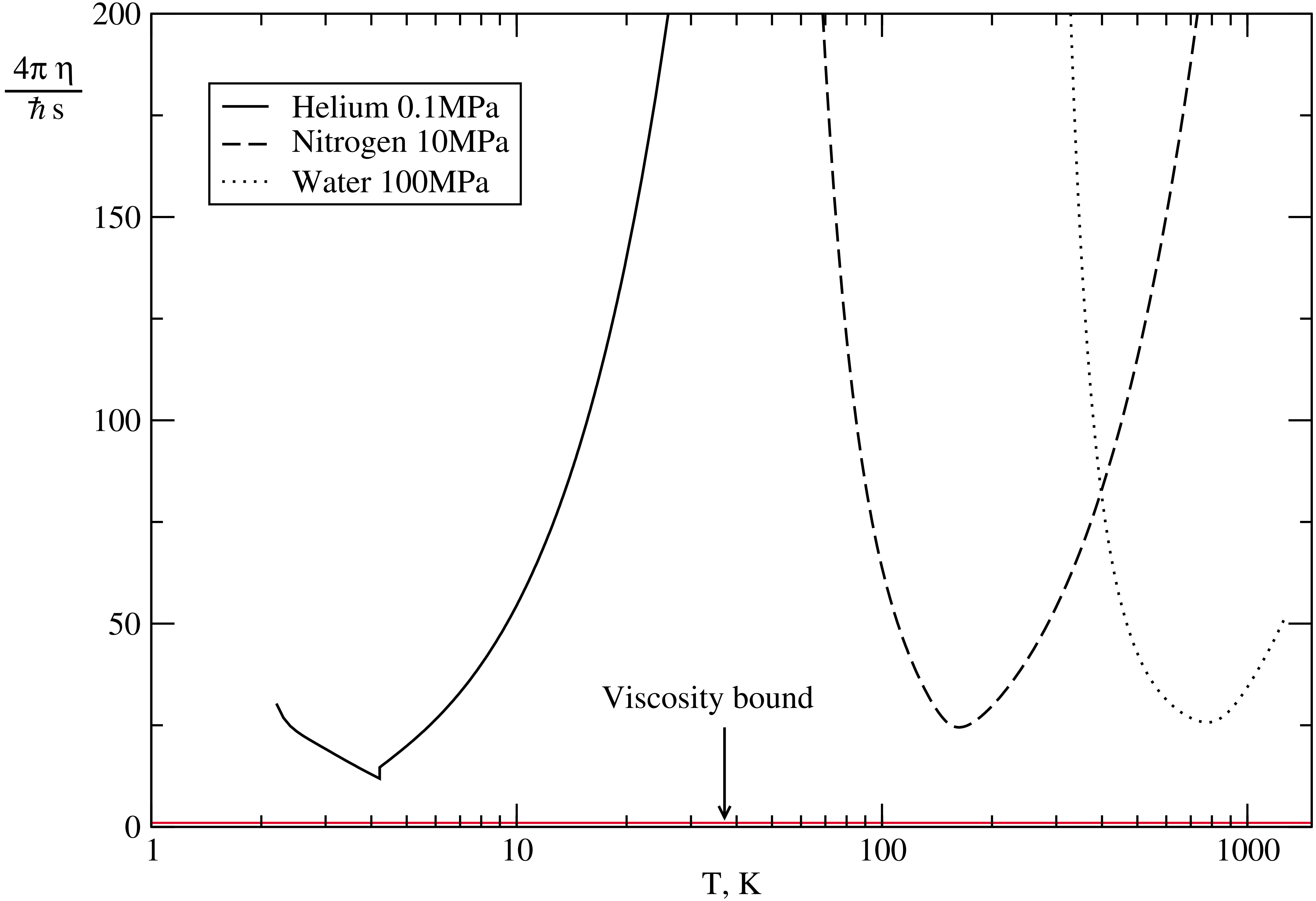
- bound is quantum mechanical
  ➡ need quantum-mechanical scattering mechanism

- bound is incompatible with weak coupling and kinetic theory:

$$\hbar/\tau \ll \epsilon_{\rm qp} \simeq k_B T\,, \quad s \sim n k_B \;: \qquad \frac{\eta}{s} \sim \frac{\epsilon_{\rm qp}\tau}{k_B} \gg \frac{\hbar}{k_B}$$

➡ need strong interactions, no good quasi-particles
(above Quantum Critical Point: $\hbar/\tau = \mathcal{C} k_B T$ [Sachdev])

Technische Universität München

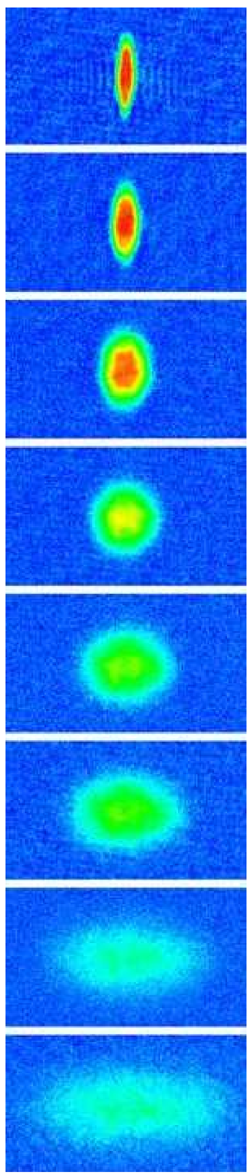


viscosity for helium, nitrogen, and water

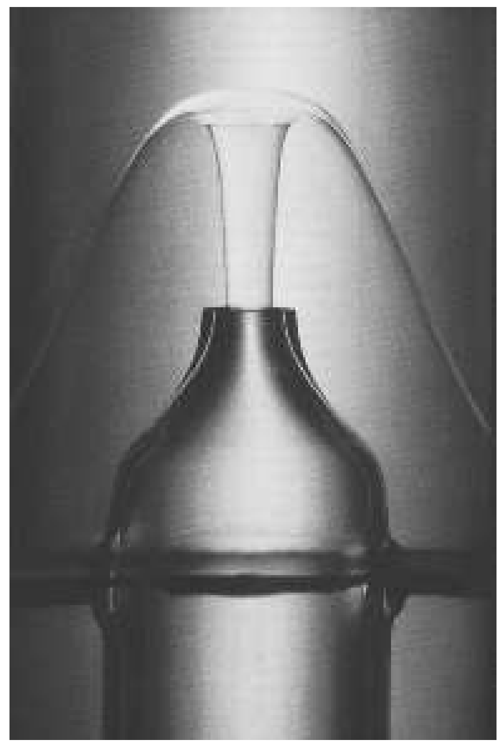
[Kovtun, Son, Starinets 2005]

Perfect fluids: the contenders

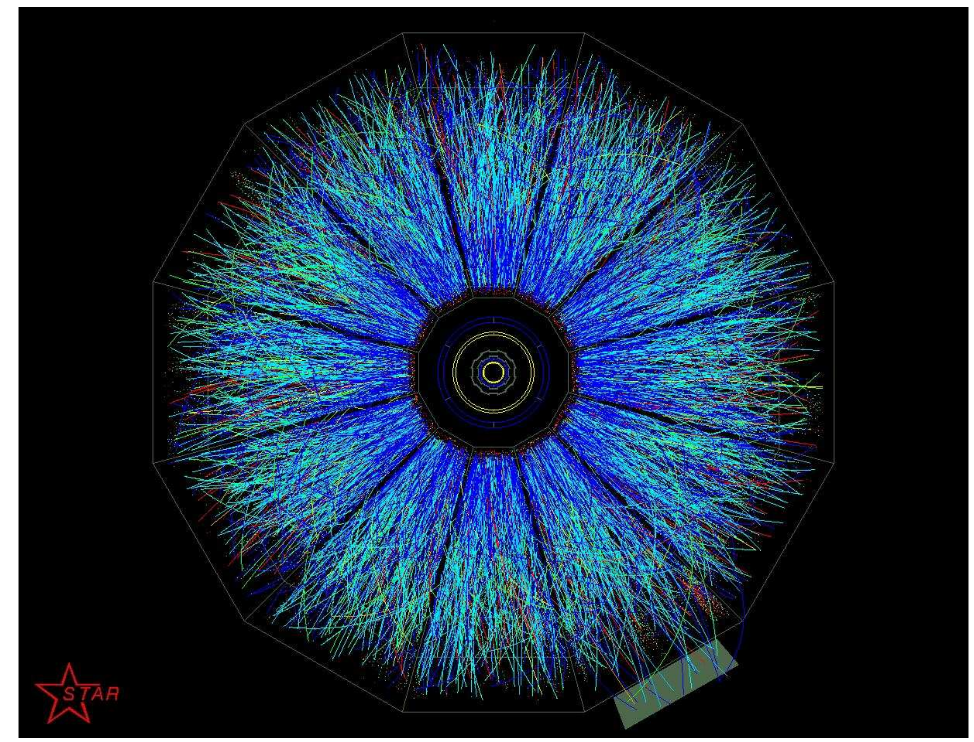
[Schaefer, Teaney 2009]



Trapped Atoms
(T=0.1 neV)
$\eta = 1.7 \cdot 10^{-15}\,\mathrm{Pa \cdot s}$



Liquid Helium
(T=0.1 meV)
$\eta = 1.7 \cdot 10^{-6}\,\mathrm{Pa \cdot s}$



Quark-Gluon Plasma
(T=180 MeV)
$\eta = 5 \cdot 10^{11}\,\mathrm{Pa \cdot s}$

Consider ratios η/s
min=0.5; 0.8; 0.4

Technische Universität München

The unitary Fermi gas

- two-component Fermi gas with zero-range interactions:

$$\mathcal{L}_E = \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger \left(\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 \right) \psi_\sigma + \frac{g(\Lambda)}{2} \psi_\sigma^\dagger \psi_{-\sigma}^\dagger \psi_{-\sigma} \psi_\sigma$$

- renormalized coupling $g(\Lambda) \mapsto g = 4\pi\hbar^2 \boldsymbol{a}/m$

- ultracold atoms: interaction range $r_0 \ll k_F^{-1} \ll \boldsymbol{a}$ scattering length

- Hubbard-Stratonovich transformation with pair field ϕ

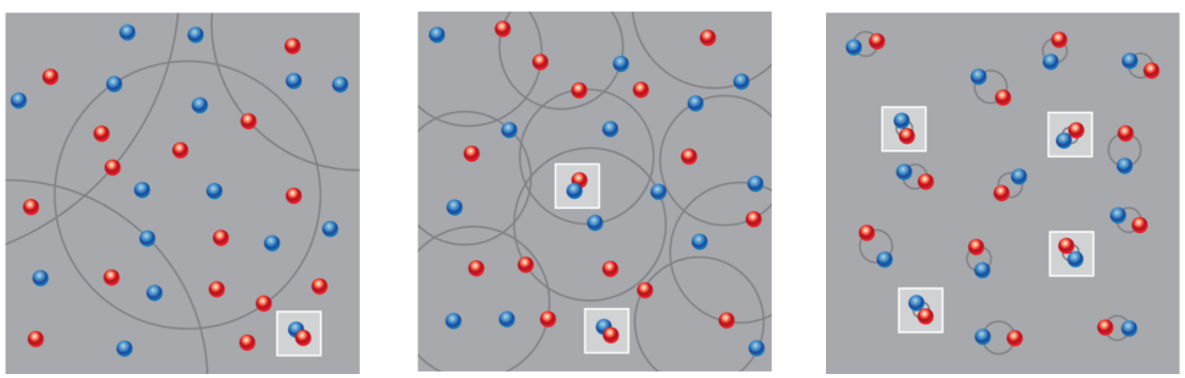
$$\mathcal{L}[\psi] \mapsto \mathcal{L}[\psi,\phi] = \mathcal{L}_0 + \left(\psi_\uparrow \psi_\downarrow \phi^* + \mathrm{h.c.} \right) - \frac{1}{g}\,\phi^*\phi$$

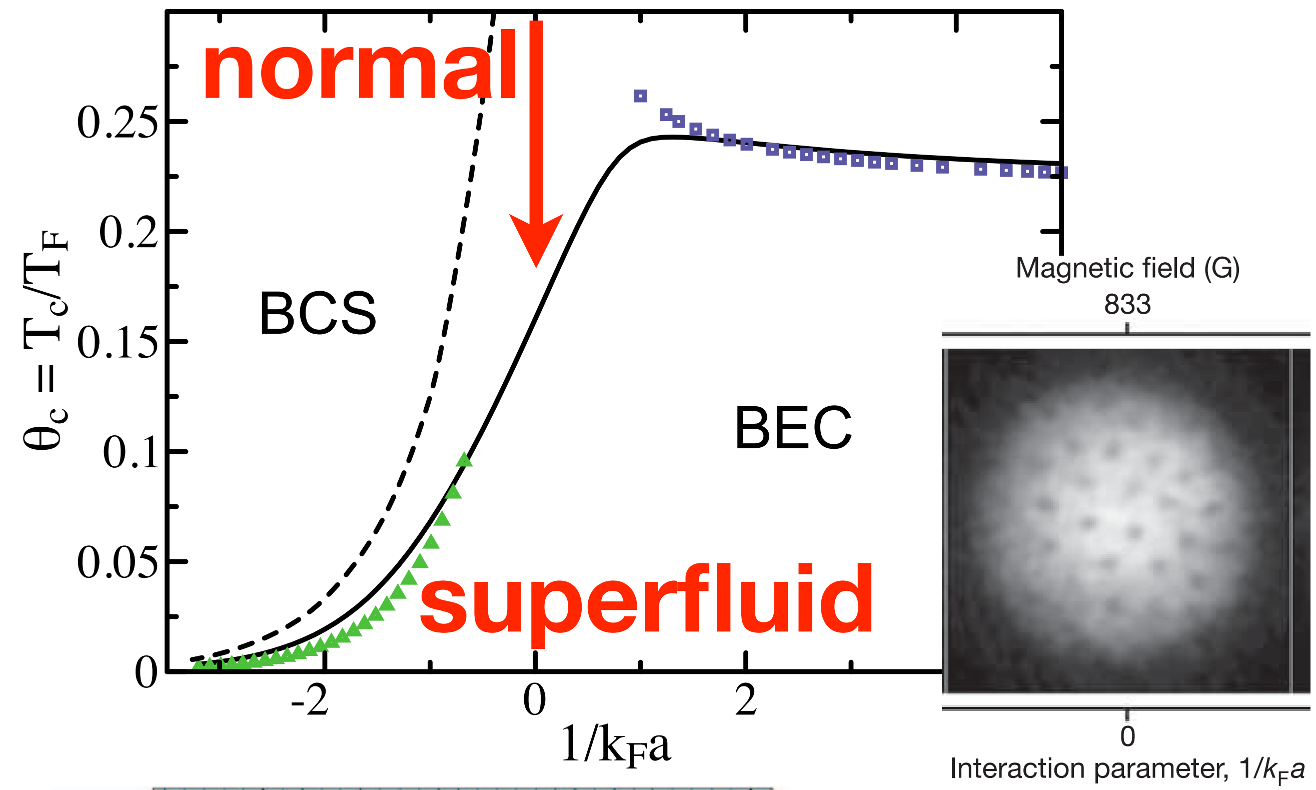
- ϕ is massless at infinite coupling g=∞ (unitarity)

Technische Universität München

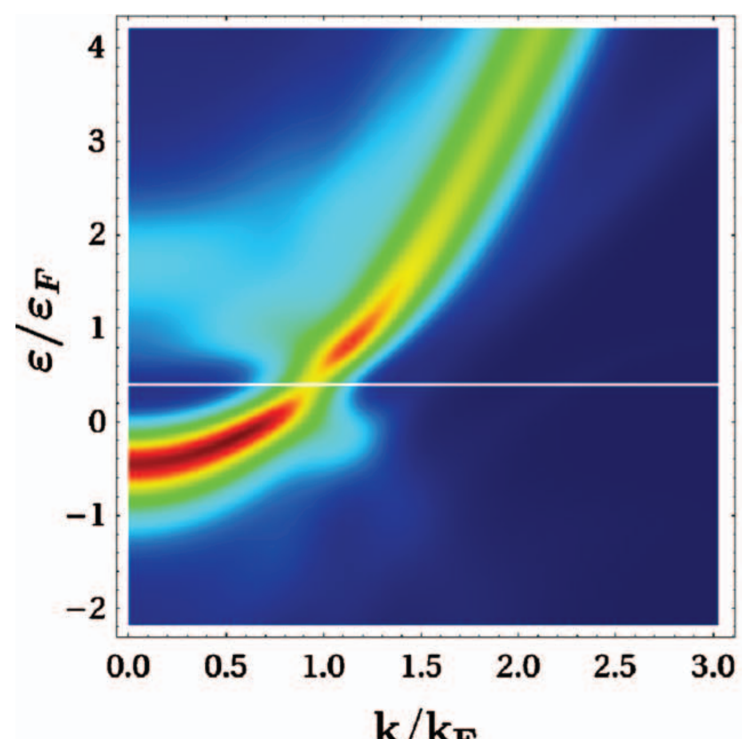
The unitary Fermi gas

- unitary Fermi gas is superfluid below $T_c \approx 0.15\, T_F$





- single-particle properties: spectral function A(k,ε) at Tc from self-consistent T-matrix [Haussmann, Punk, Zwerger 2009]



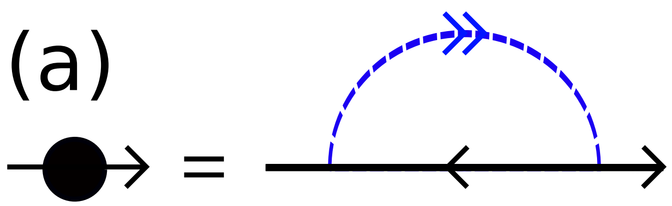
- now two-particle properties: transport!

sche Universität München

Baym-Kadanoff (2PI) conserving approximation

- self-consistent equations for fermionic and bosonic self-energies:

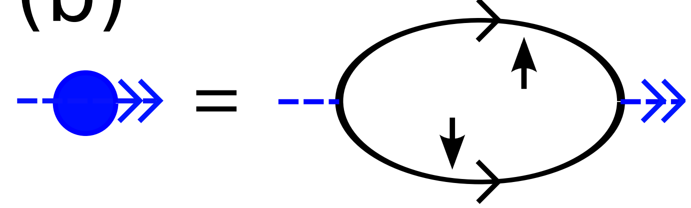
(a)



$$\Sigma_\uparrow(K) = \sum_Q \Gamma(Q) G_\downarrow(Q+K)$$

$$\Sigma_\uparrow(X) = \Gamma(X) G_\downarrow(-X)$$

(b)



$$\Sigma_b(Q) = \sum_K G_\uparrow(K) G_\downarrow(Q-K)$$

$$\Sigma_b(X) = G_\uparrow(X) G_\downarrow(X)$$

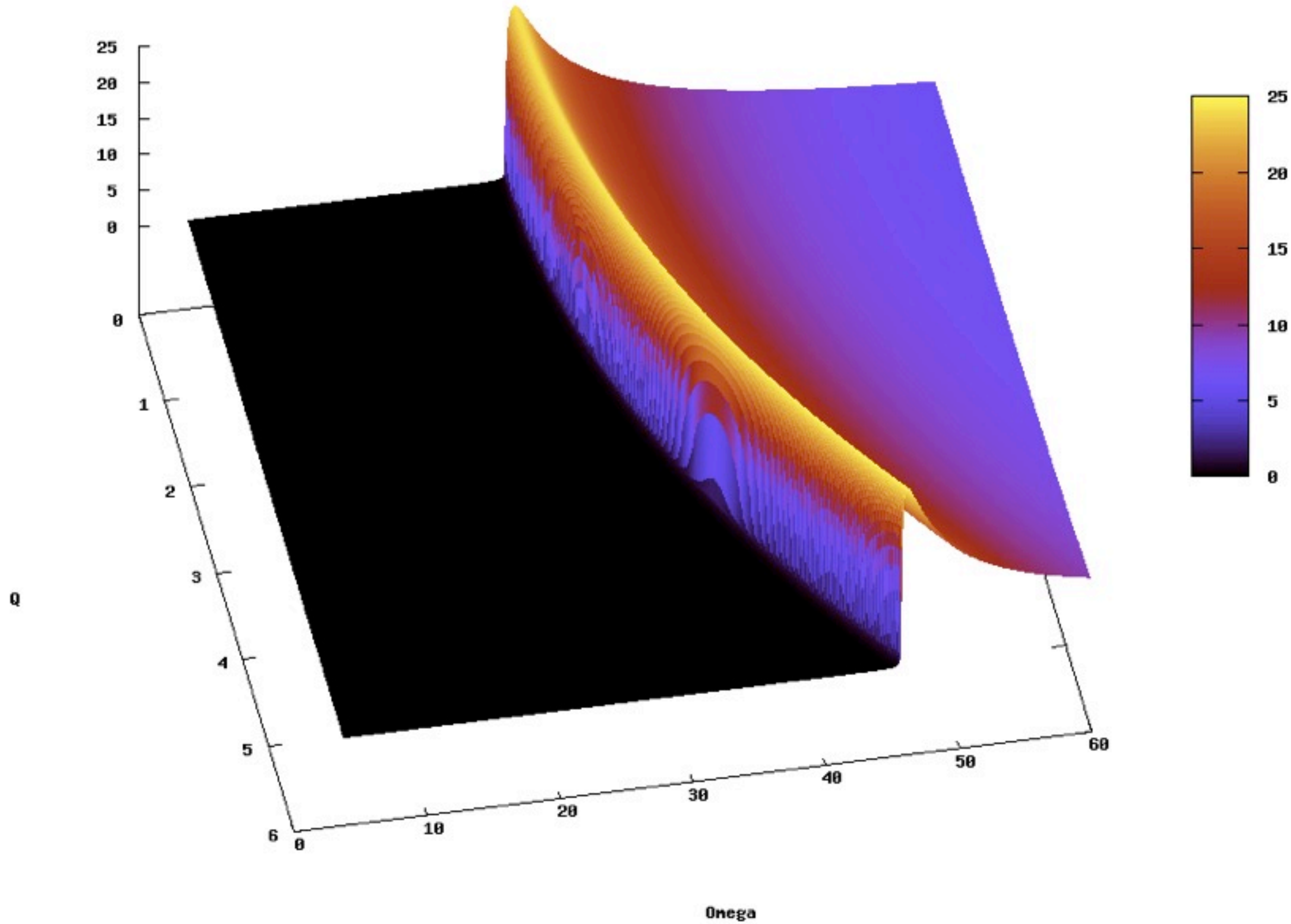
- Dyson equation:

$$G_\uparrow^{-1}(K) = G_0^{-1}(K) - \Sigma_\uparrow(K) \qquad \Gamma^{-1}(Q) = a^{-1} - \Sigma_b(Q)$$

- solve by iteration on logarithmic grid (300 frequencies & 300 radial momenta)

- conserving: number, momentum current, scale invariance, ...
  fulfills Tan energy formula & adiabatic relation exactly

Technische Universität München



Pair spectral function

Viscosity in linear response: Kubo formula

- viscosity from stress correlations (cf. hydrodynamics):

$$\eta(\omega) = \frac{1}{\omega} \int d^3x\, dt\, e^{i\omega t}\, \theta(t) \left\langle \left[\hat{\Pi}_{xy}(\vec{x}, t), \hat{\Pi}_{xy}(0,0)\right] \right\rangle$$

with stress tensor $\hat{\Pi}_{xy} = \sum_{\mathbf{p},\sigma} \frac{p_x p_y}{m}\, c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma}$ (cf. Newton $\frac{\partial v_x}{\partial y}$)

- or, from transverse current correlations [Hohenberg, Martin 1965]

$$\chi_\perp(\omega, \mathbf{q}) = \frac{\eta q^2}{\omega^2 + (D_\perp q^2)^2}, \qquad \eta = D_\perp \rho_{\mathrm{n}}$$

- equivalence via momentum balance

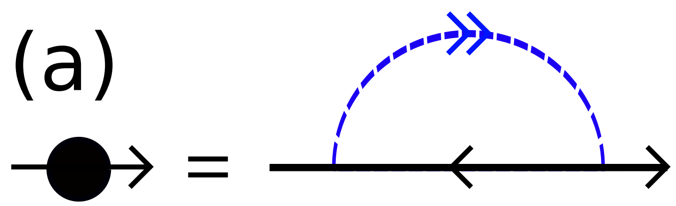
$$\partial_t(\rho v_i) + \partial_j \Pi_{ij} = 0$$

Transport equations

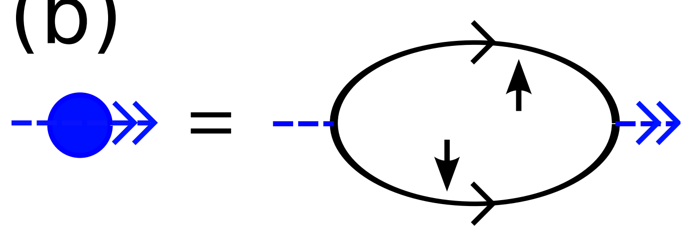
[Enss, Haussmann, Zwerger 2010]

- Single-particle Green functions:

(a)

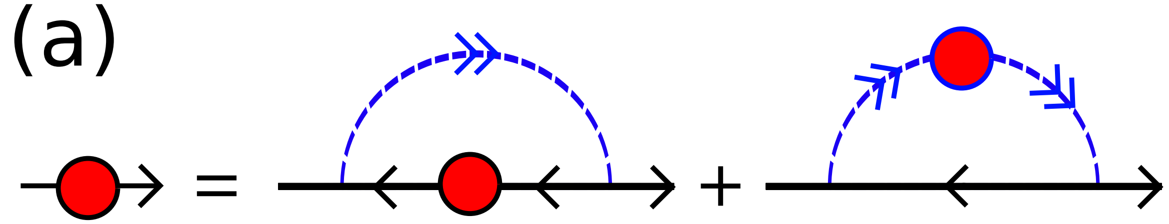


(b)

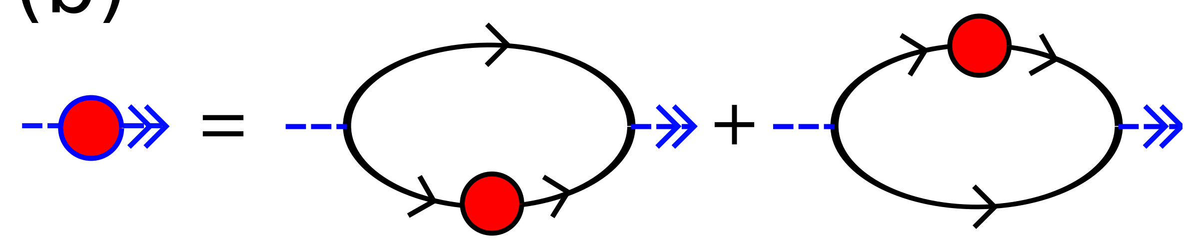


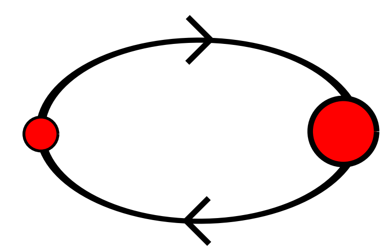
Response to shear perturbations:

(a)



(b)

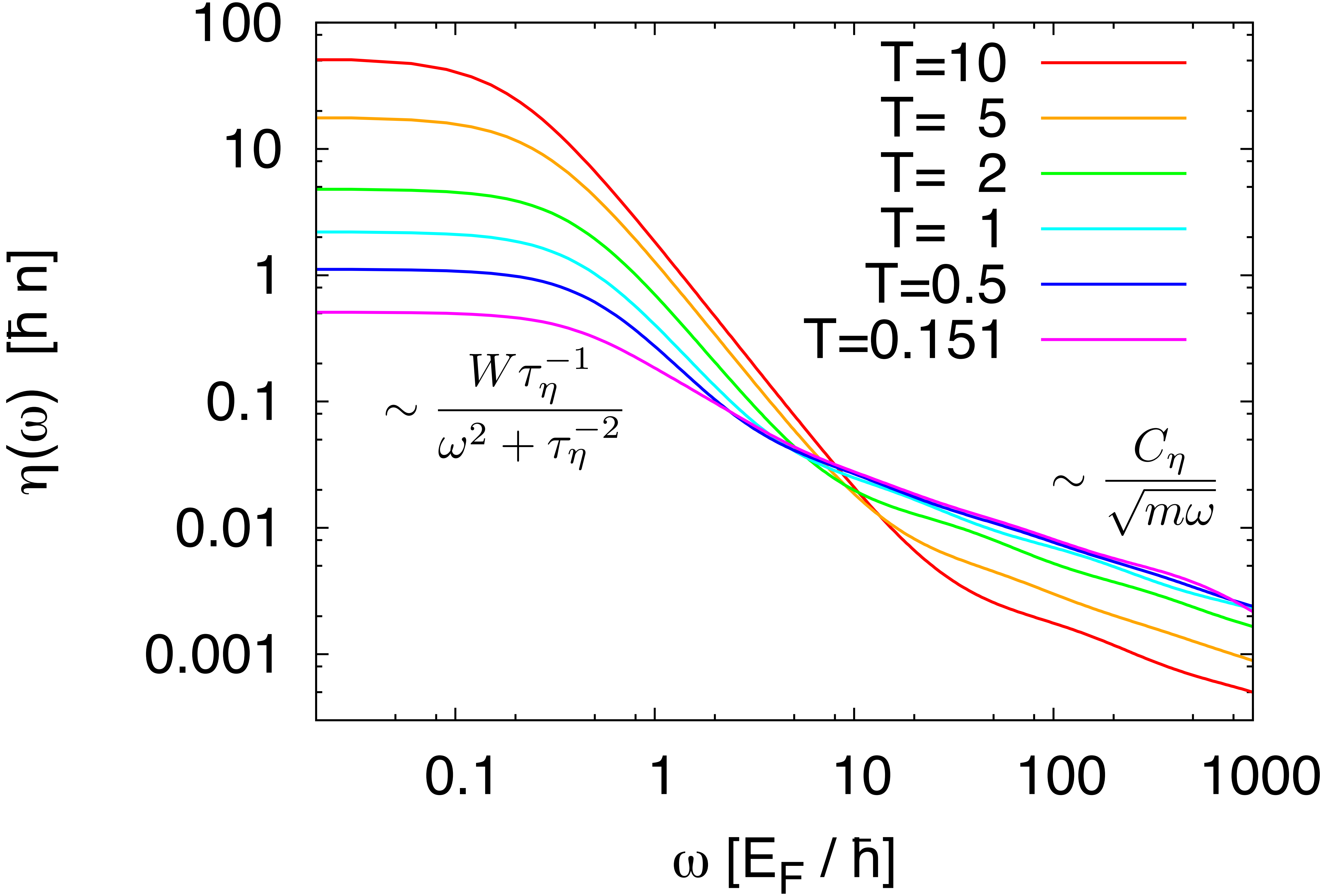


- correlation function (Kubo formula): $\eta(\omega) =$ 

- transport via fermionic and bosonic modes:
  very efficient description, satisfies conservation laws

- assumes no quasiparticles: beyond Boltzmann

Technische Universität München



Viscosity spectral function

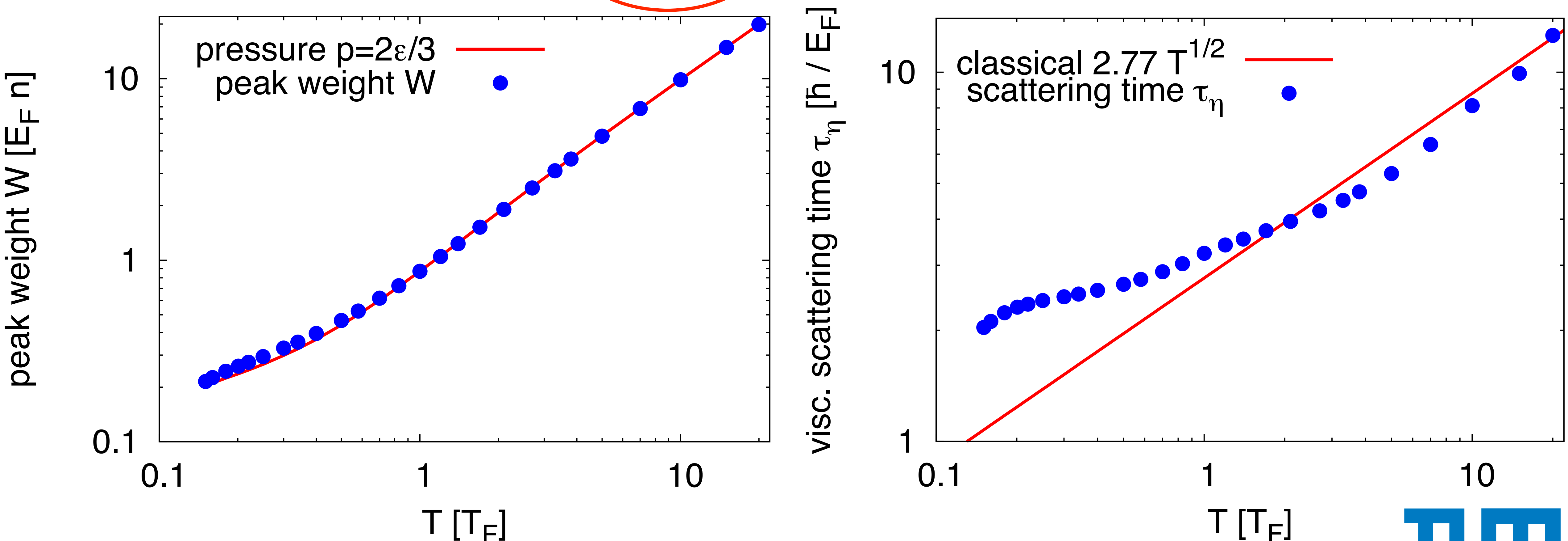
[Enss, Haussmann, Zwerger 2010]

Transport peak

[Enss, Haussmann, Zwerger 2010]

- fit to Drude peak and tail:

$$\eta(\omega) = \frac{W\tau_\eta^{-1}}{\omega^2 + \tau_\eta^{-2}} + \frac{\hbar^{3/2}C_\eta}{\sqrt{m\omega}}\ \frac{\omega(\omega + \tau^{-1})}{\omega^2 + \tau^{-2}}$$



Technische Universität München

- near Tc: $\hbar/\tau_\eta \sim 0.5\, E_F \sim 3\, k_B T$
  decay rate larger than energy, no good quasiparticles!

Contact coefficient

- generically, short-distance (UV) behavior depends on non-universal details of interaction potential

- for zero-range interaction ($r_0 \ll k_F^{-1}$) this becomes universal:
  at most two particles within distance r_0, all others far away (medium)

- two-particle density matrix for $r_0 < r \ll k_F^{-1}$: **many-body** **few-body**

$$\int d^3\mathbf{R}\,\left\langle \psi^\dagger_\uparrow(\mathbf{R}+\frac{\mathbf{r}}{2})\psi^\dagger_\downarrow(\mathbf{R}-\frac{\mathbf{r}}{2})\psi_\downarrow(\mathbf{R}-\frac{\mathbf{r}}{2})\psi_\uparrow(\mathbf{R}+\frac{\mathbf{r}}{2})\right\rangle = C\left(\frac{1}{r}-\frac{1}{a}\right)^2$$

- Tan contact C: probability of finding up and down close together (property of strongly coupled medium)
  [Tan 2005; Braaten, Platter 2008; Zhang, Leggett 2009; Combescot et al. 2009; Haussmann et al. 2009; Schneider, Randeria 2010; Son, Thompson 2010; Werner, Castin; Braaten 2010...]

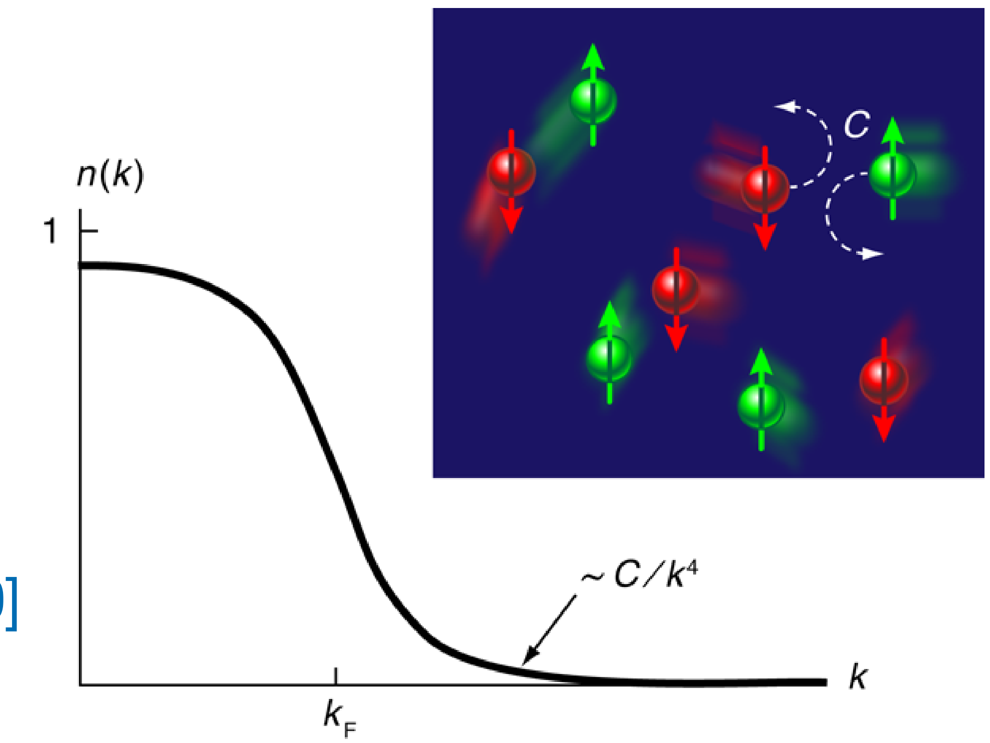
Technische Universität München

Contact coefficient (2)

- determine C:

$$\lim_{p\to\infty} n_p = \frac{C}{p^4}$$

[Stewart et al. 2010]



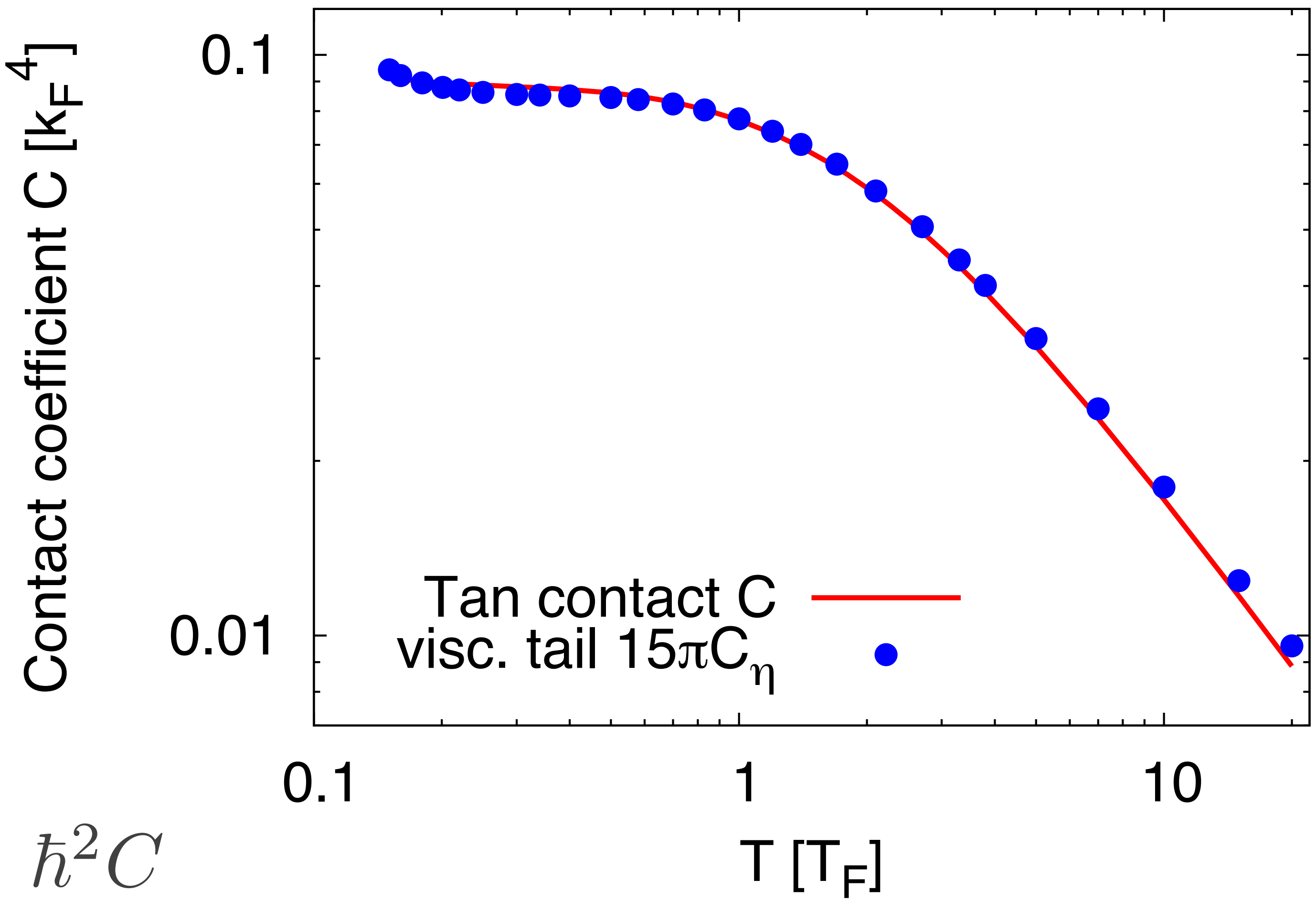
- **intuitively:** absorb external perturbation with large energy/momentum far away from coherent peak of a single particle
  ➡ need to hit 2 particles close together to give energy+momentum to both
  ➡ absorption rate ~C

- access strong coupling and arbitrary temperature via perturbation theory

Viscosity tail

- analytical expression for tail
  [Enss, Haussmann, Zwerger 2010]

$$\eta(\omega \to \infty) = \frac{\hbar^{3/2} C}{15\pi\sqrt{m\omega}}$$

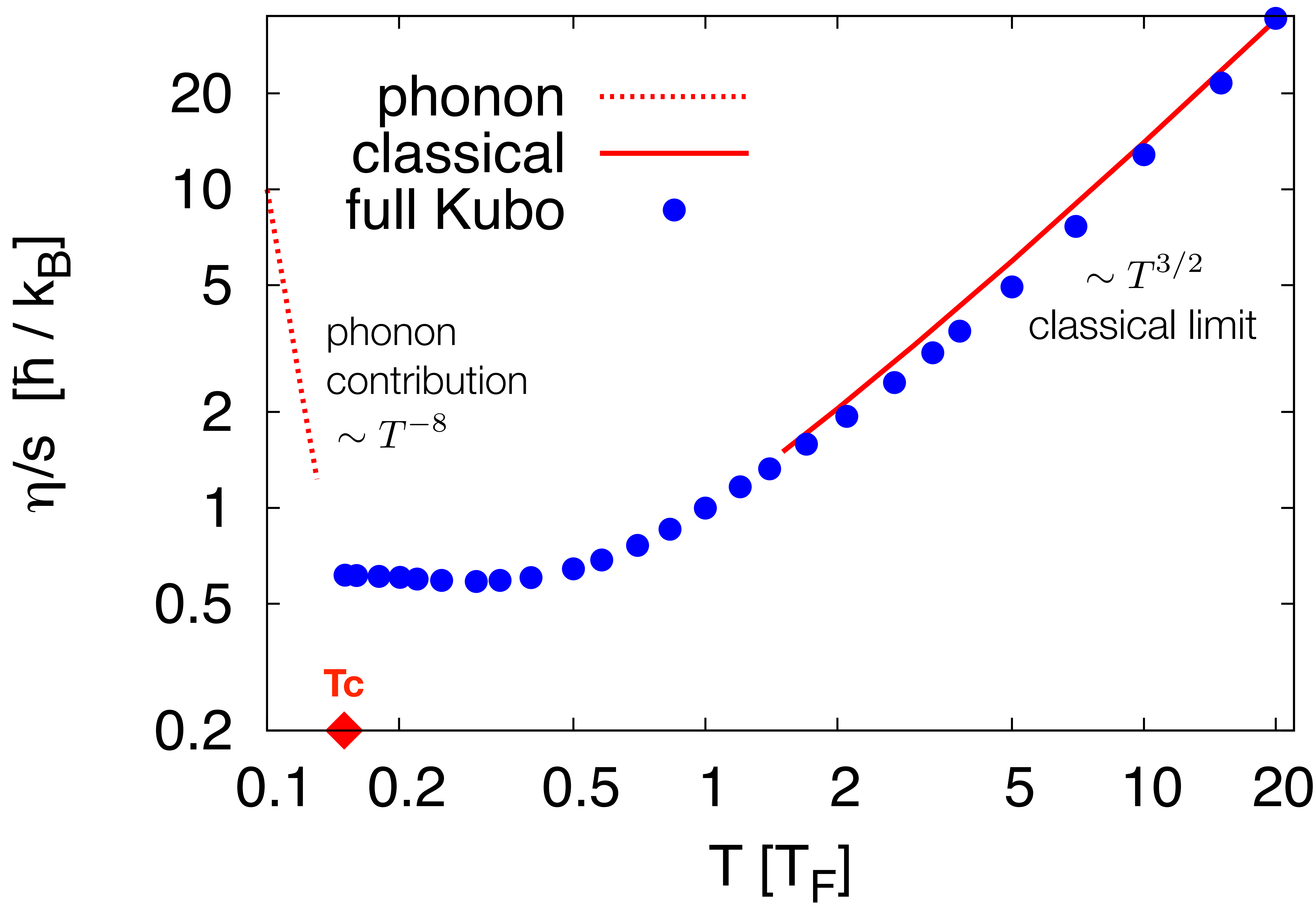


- viscosity sum rule

$$\frac{2}{\pi}\int_0^\infty d\omega\ [\eta(\omega) - \text{tail}] = p - \frac{\hbar^2 C}{4\pi m a}$$

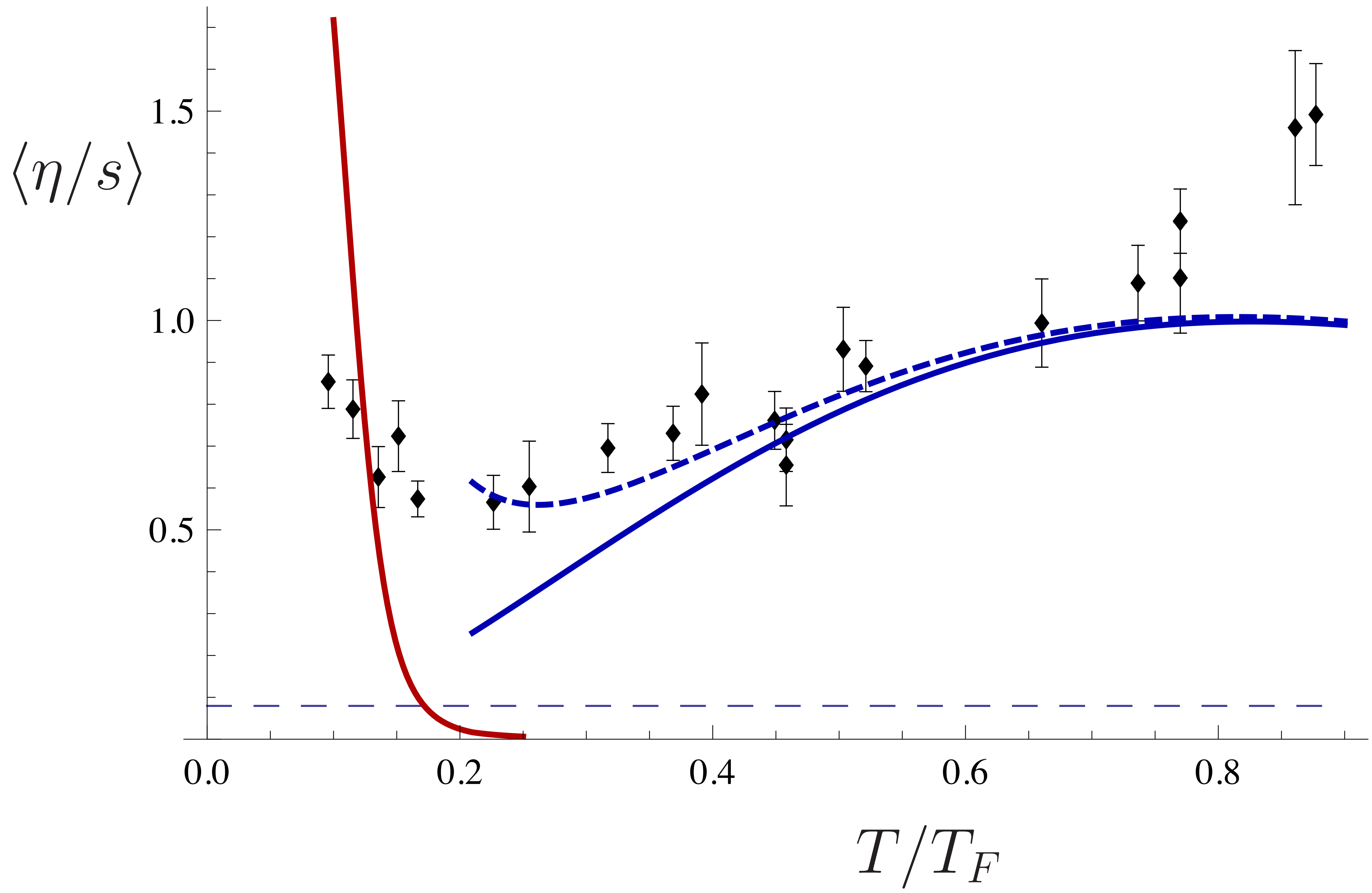
provides non-perturbative check
[Enss, Haussmann, Zwerger 2010;
cf. Taylor, Randeria 2010]

Technische Universität München



Shear viscosity/entropy
of the unitary Fermi gas

[Enss, Haussmann, Zwerger 2010]



shear viscosity
for trapped atoms

[fit: Schaefer, Chafin 2009;
exp: Kinast et al. 2004, Luo et al. 2009]

Scale invariance

- Unitary Fermi gas: particle spacing only length scale, **scale invariance**
- pressure $3\hat{p} = \int d^3x\, \hat{\Pi}_{ii}(\mathbf{x},t) = 2\hat{H} = 2(\hat{T}+\hat{V})$ ➡ $3p = 2\epsilon$ [Ho 2004]
- bulk viscosity $\zeta(\omega) = \frac{1}{9\omega}\int d^3x\, dt\, e^{i\omega t}\,\theta(t)\left\langle\left[\hat{\Pi}_{ii}(\mathbf{x},t),\hat{\Pi}_{jj}(0,0)\right]\right\rangle \equiv 0$
  [Nishida, Son 2007]
- response of Green's functions to scale transformation: **Ward identities**
  [Enss, Haussmann, Zwerger 2010]

$$\omega\mathcal{T}[\Pi_{ii}] = (\epsilon+\mu)G(p,\epsilon) - (\epsilon+\omega+\mu)G(p,\epsilon+\omega)$$
$$\omega\mathcal{T}_{\text{bos}}[\Pi_{ii}] = (\Omega+\omega+2\mu)\Gamma(p,\Omega) - (\Omega+2\mu)\Gamma(p,\Omega+\omega)$$

- WI provides solution of transport equation:
  conserving approximation

Technische Universität München

Beyond viscosity: universal dynamics

- viscosity determines several correlation functions at unitarity (small q):
  [cf. Taylor, Randeria 2010]

mom. conserv.

$$\mathrm{Im}\,\langle[\Pi_{xy},\Pi_{xy}]\rangle_{q,\omega} = \omega\,\eta(\omega) \qquad \text{(generally)}$$

$$\mathrm{Im}\,\langle[J_i,J_k]\rangle^{T}_{q,\omega} = \frac{q^2}{\omega}\,\eta(\omega) \qquad \text{(generally)}$$

at unitarity

$$\mathrm{Im}\,\langle[J_i,J_k]\rangle^{L}_{q,\omega} = \frac{q^2}{\omega}\left(\frac{4}{3}\,\eta(\omega) + \cancel{\zeta(\omega)}\right) \qquad \text{(generally)}$$

charge conserv.

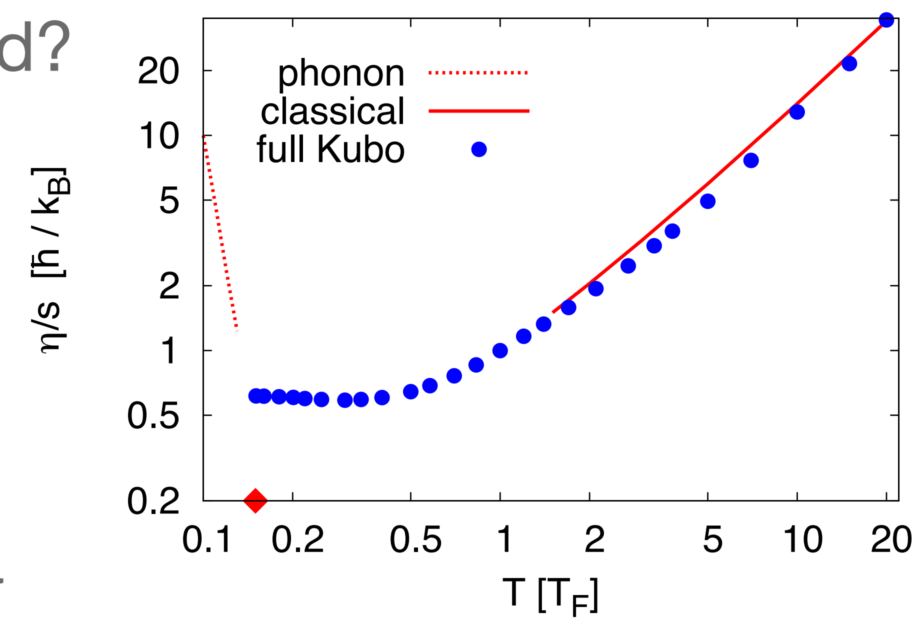
$$\mathrm{Im}\,\langle[\rho,\rho]\rangle_{q,\omega} = \frac{q^4}{\omega^3}\,\frac{4}{3}\,\eta(\omega) \qquad \text{(at unitarity)}$$

$$\langle[J_i,J_k]\rangle = \frac{q_iq_k}{q^2}\langle[J_i,J_k]\rangle^{L} + \left(\delta_{ik} - \frac{q_iq_k}{q^2}\right)\langle[J_i,J_k]\rangle^{T}$$

Conclusion and Outlook

Technische Universität München

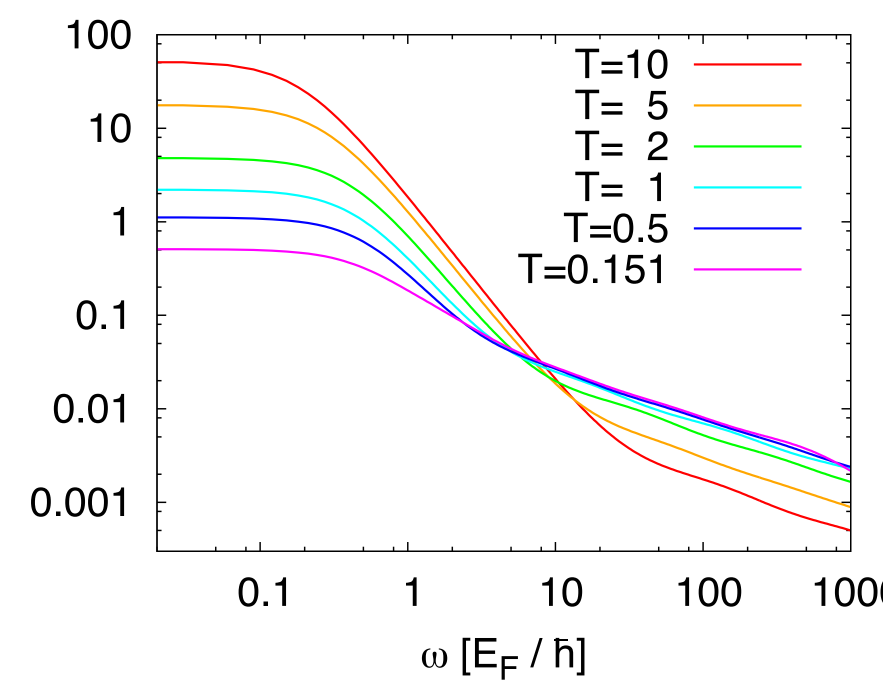
- are cold atoms (unitary Fermi gas) the perfect fluid?



  ➡ most perfect real non-relativistic fluid known (factor ≈7 above string theory bound)

  ➡ strongest contender with quark-gluon plasma

- transport calculation of η(ω) beyond Boltzmann (tail, large scattering rate)



- conserves **scale invariance** and fulfills sum rules

- universal **dynamic** properties of unitary Fermi gas